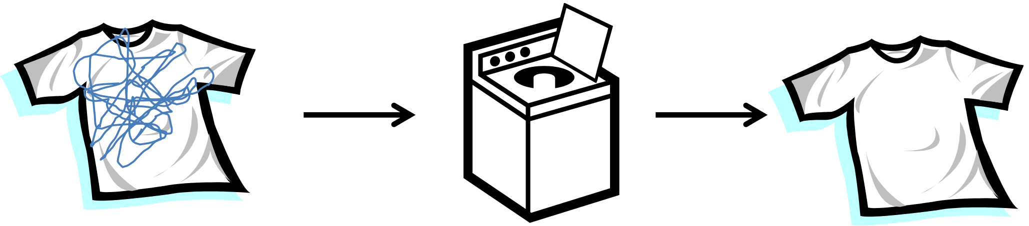
Definition: Function Notation

Function notation is defined to be of the form:

$$FUNCTION\ NAME(INPUT) = DESCIPTION\ ON\ THE\ OUTPUT$$

Example 1:

Determine the name, input, and a description of the output of the function given below. Then use function notation.



Solution:

Name of Function:	Washing Machine
Input:	Shirt
Output:	Clean
Function notation:	Washing_Machine(shirt) = clean shirt

Function Notation Examples

Example 2:
Determine the name, input, and a description of the output of the function $f(x) = x^2$

Solution:

Name of Function:	f
Input:	x
Output:	x^2

Example 3:
Determine the value(s) of x where $g(x) = f(x)$ and $f(x) = 3x^2 + 2$ and $g(x) = 4x + 1$

Solution:

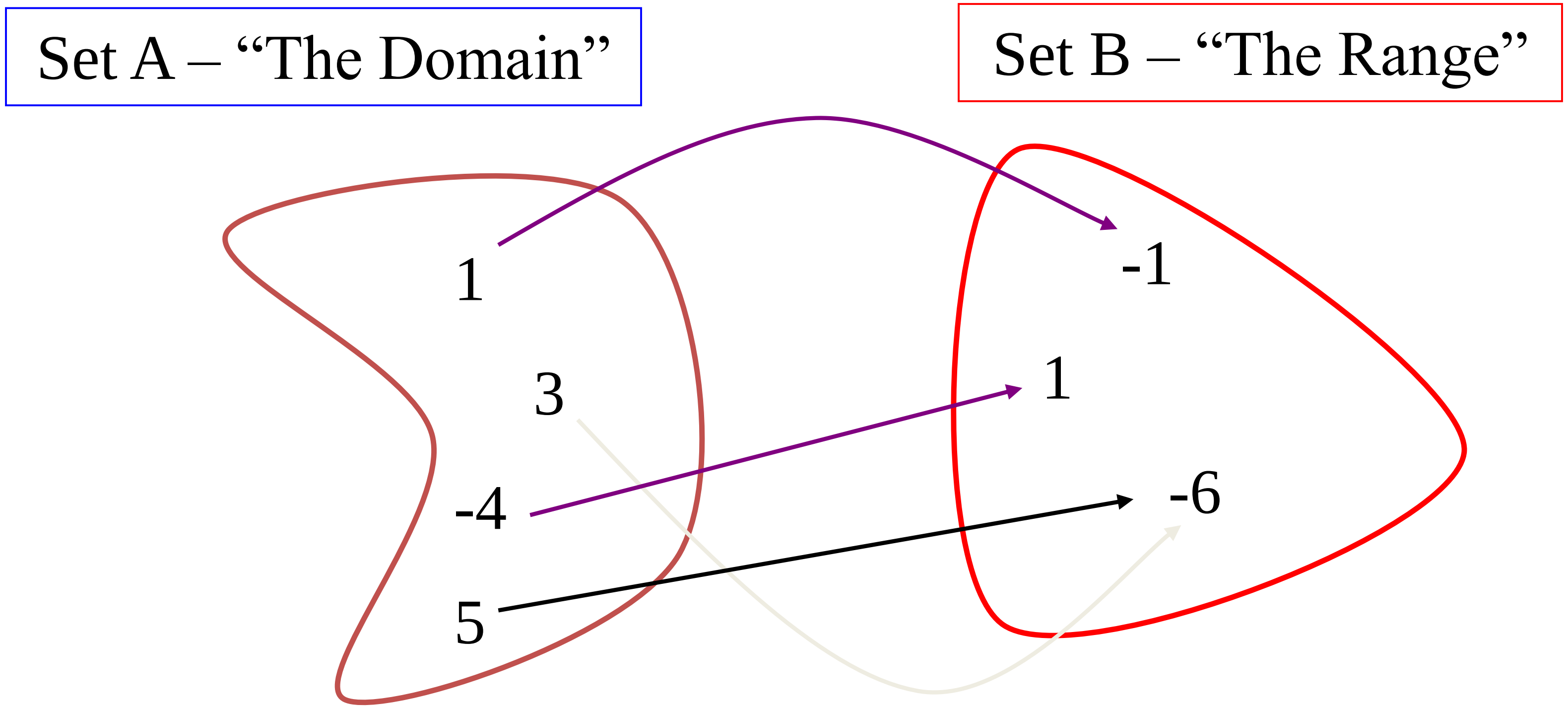
$$\begin{aligned} f(x) &= g(x) \\ 3x^2 + 2 &= 4x + 1 \\ 3x^2 - 4x + 1 &= 0 \\ (3x - 1)(x - 1) &= 0 \end{aligned}$$

$$\therefore 3x - 1 = 0 \; or \; x - 1 = 0$$

$$\therefore x = \frac{1}{3} \; or \;\; x = 1$$

Functions

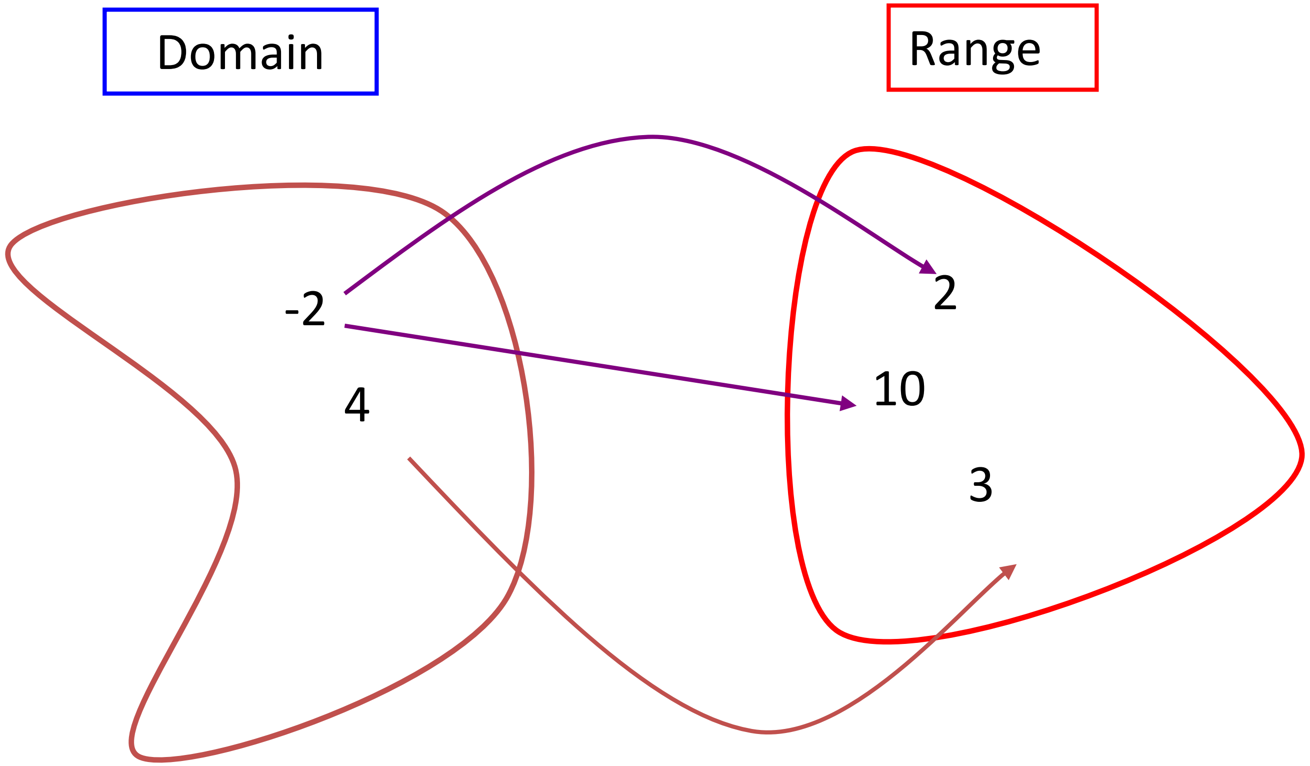
A **function** is a rule that assigns to each element in a set A one and only one element in a set B.



NOTE: We can use each element in the range more than once.

Functions

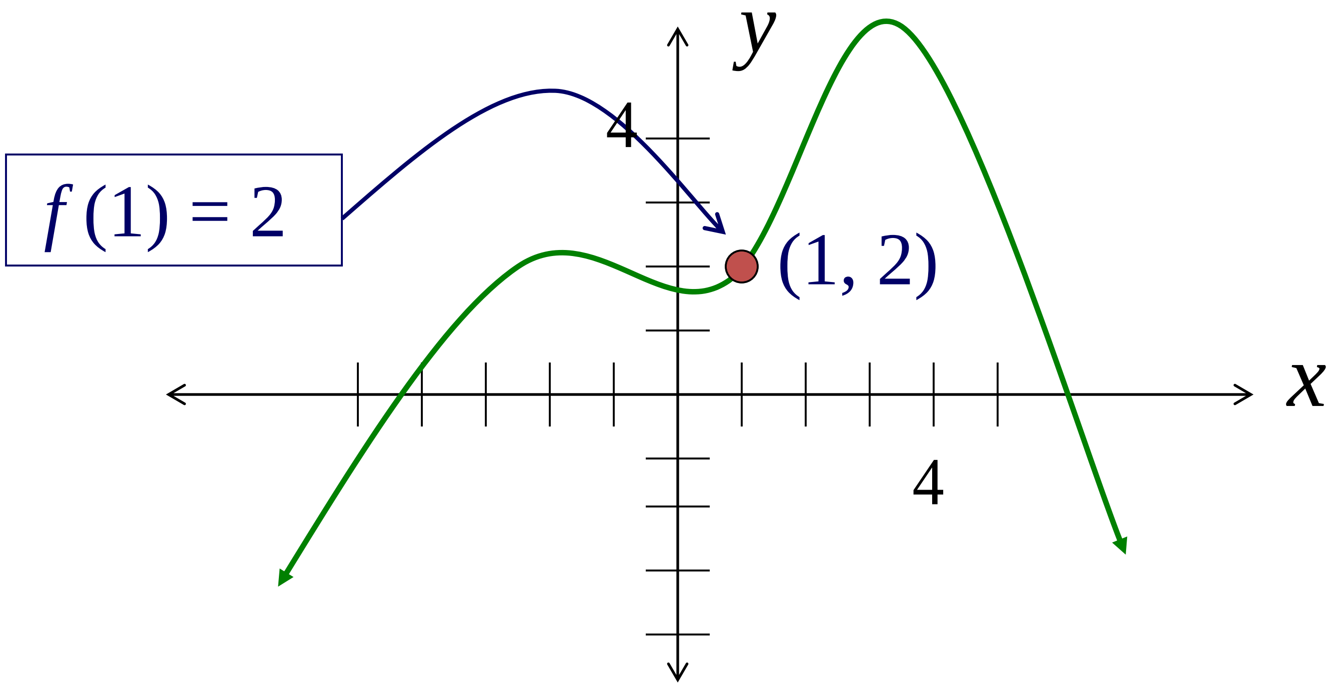
We call a function a mapping from one set to another set in such a way that each input will only produce one output. The following diagram does not represent a function.



Why not? There is one element in the domain (-2) assigned to two elements in the range (2 and 10).

Graph of a Function

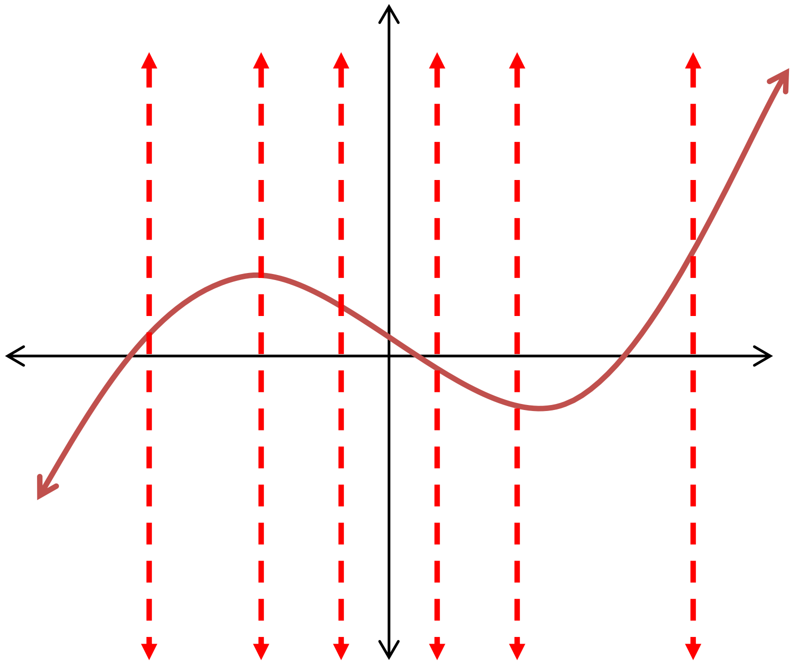
The **graph** of a function f is the set of all ordered pairs (x, y) in the the xy–plane such that x is in the domain of f and $y = f(x)$. *x is called the independent variable and y is called the value of the function.*



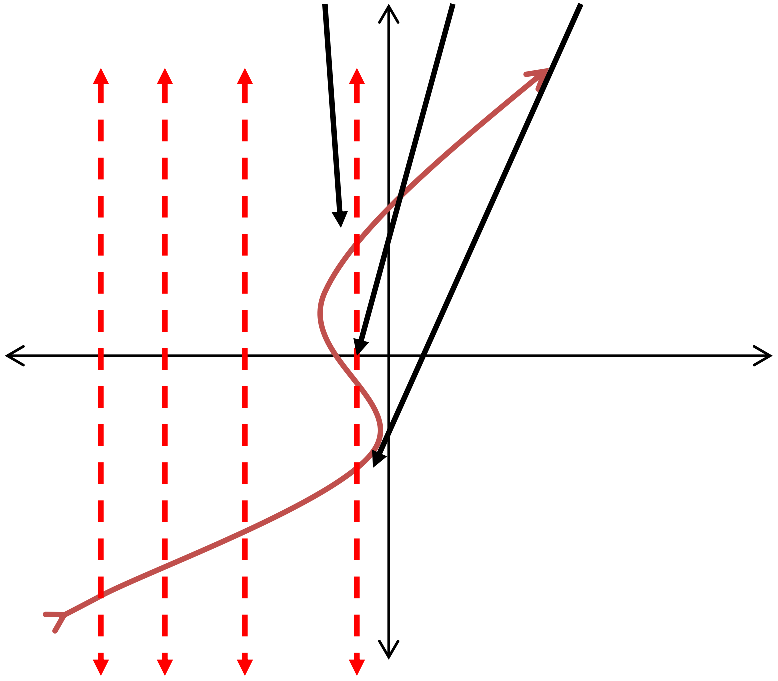
Graph of a Function

Vertical Line Test: The graph of a function can be crossed at most once by any vertical line.

Function



Not a Function



Strategy: Evaluating Functions using the Box Method

How To Use it:	When To Use it:	Why this works?
Rather than writing x (or whatever the input variable happens to be) you can instead place it with a "box" $\square$ anywhere you see x.	When evaluating any function, but typically it is helpful when you have challenging inputs.	This is simply changing the variable symbol from x to $\square$. So there is not actual change in the math (but may help to visualize the idea).

Example 4:
Determine $f(3)$ when $f(x) = x^2 + 2x + 3$

Solution:

$$f(x) \qquad = x^2 + 2x + 3$$
$$f(\square) \qquad = \square^2 + 2\square + 3$$

Place 3 inside our box to get:

$$\begin{aligned} f(3) &= 3^2 + 2(3) + 3 \\ &= 9 + 6 + 3 \\ &= 18 \end{aligned}$$

Box Method Examples

Example 5:
Determine $f(h+2)$ when $f(x) = x^2 + 2x + 3$

Solution:

$$
\begin{aligned}
f(x) &= x^2 + 2x + 3 \\
f(\square) &= \square^2 + 2\square + 3
\end{aligned}
$$

Place $h + 2$ inside our box to get:

$$
\begin{aligned}
f(h+2) &= [h+2]^2 + 2[h+2] + 3 \\
&= [h+2][h+2] + 2h + 4 + 3 \\
&= h^2 + 4h + 4 + 2h + 7 \\
&= h^2 + 6h + 11
\end{aligned}
$$

Example 6:
Determine $f(x+h) - f(x)$ when $f(x) = 3x^2 + 2$

Solution:

$$
\begin{aligned}
f(x) &= 3x^2 + 2 \\
f(\square) &= 3\square^2 + 2
\end{aligned}
$$

Place $x + h$ inside our box to get:

$$
\begin{aligned}
f(x+h) &= 3[x+h]^2 + 2 \\
&= 3[x+h][x+h] + 2 \\
&= 3(x^2 + 2xh + h^2) + 2 \\
&= 3x^2 + 6xh + 3h^2 + 2
\end{aligned}
$$

$$
\begin{aligned}
\therefore f(x+h) - f(x) &= [3x^2 + 6xh + 3h^2 + 2] - [3x^2 + 2] \\
&= 3x^2 + 6xh + 3h^2 + 2 - 3x^2 - 2 \\
&= 6xh + 3h^2
\end{aligned}
$$

Algebra of Functions

The sum, difference, product and quotient of two functions f and g is defined as follows:

$$(f \pm g)(x) = f(x) \pm g(x)$$

$$(f.g)(x) = f(x).g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Definition: Composing Functions

<u>Function composition</u> occurs when we put the output of one function as the input for a second function. We use the symbol "o" to indicate that we are doing composition.

$$f\ o\ g\ (x) = f\big(g(x)\big)$$

<u>Example 7</u>:

Determine $f\ o\ g\ (x)$ and $g\ o\ f\ (x)$ when $f(x) = x^2 + 2x + 3$ and $g(x) = 2x + 1$

<u>Solution:</u>

$$\begin{aligned} f\ o\ g(x) &= f\big(g(x)\big) \\ &= [g(x)]^2 + 2[g(x)] + 3 \\ &= [2x+1]^2 + 2[2x+1] + 3 \\ &= (2x+1)(2x+1) + 4x + 2 + 3 \\ &= 4x^2 + 4x + 1 + 4x + 5 \\ &= 4x^2 + 8x + 6 \end{aligned}$$

$$\begin{aligned} g\ o\ f(x) &= g\big(f(x)\big) \\ &= 2[f(x)] + 1 \\ &= 2[x^2 + 2x + 3] + 1 \\ &= 2x^2 + 4x + 7 \end{aligned}$$

Composing Functions Examples

Example 8:
We note that in the previous example $f\ o\ g$ and $g\ o\ f$ are different. It is usually the case that the order matters for two functions. Are you able to find a pair of functions where $f\ o\ g$ and $g\ o\ f$ are the same?

Solution:

If $f(x) = g(x)$ then we would clearly have $f\ o\ g = g\ o\ f = f\ o\ f = g\ o\ g$

If $g(x) = x$ and the other function is any function $f(x)$ we would get $f\ o\ g = g\ o\ f = f(x)$

Can you think of others?

Definition: Domain and Range

Given a function, we call the **domain** the set of all allowable inputs (x values) that the function can take.

Similarly, we call the **range** the set of all allowable outputs (y values) the function can map to.

Example 1:
Watch the short clip found here: https://www.youtube.com/watch?v=8_jLpd-_gdY

Identify the function (machine):
Something that is in the domain of the function:
Something that is not in the domain of the function:
Something that is in the range of the function:
Something that is not in the range of the function:

Solution:
We have many answers, but some typical responses base on the video would be:

Identify the function (machine):	Washing Machine
Something that is in the domain of the function:	Shirt
Something that is not in the domain of the function:	Brick
Something that is in the range of the function:	Clean Shirt
Something that is not in the range of the function:	Clean Brick (or brick)

Finding the domain

The domain of sum, difference and product of two functions f and g are the set of all x in the domain of **both** f and g.

The domain of the quotient $\frac{f}{g}$ excludes all the x such that $g(x) = 0$

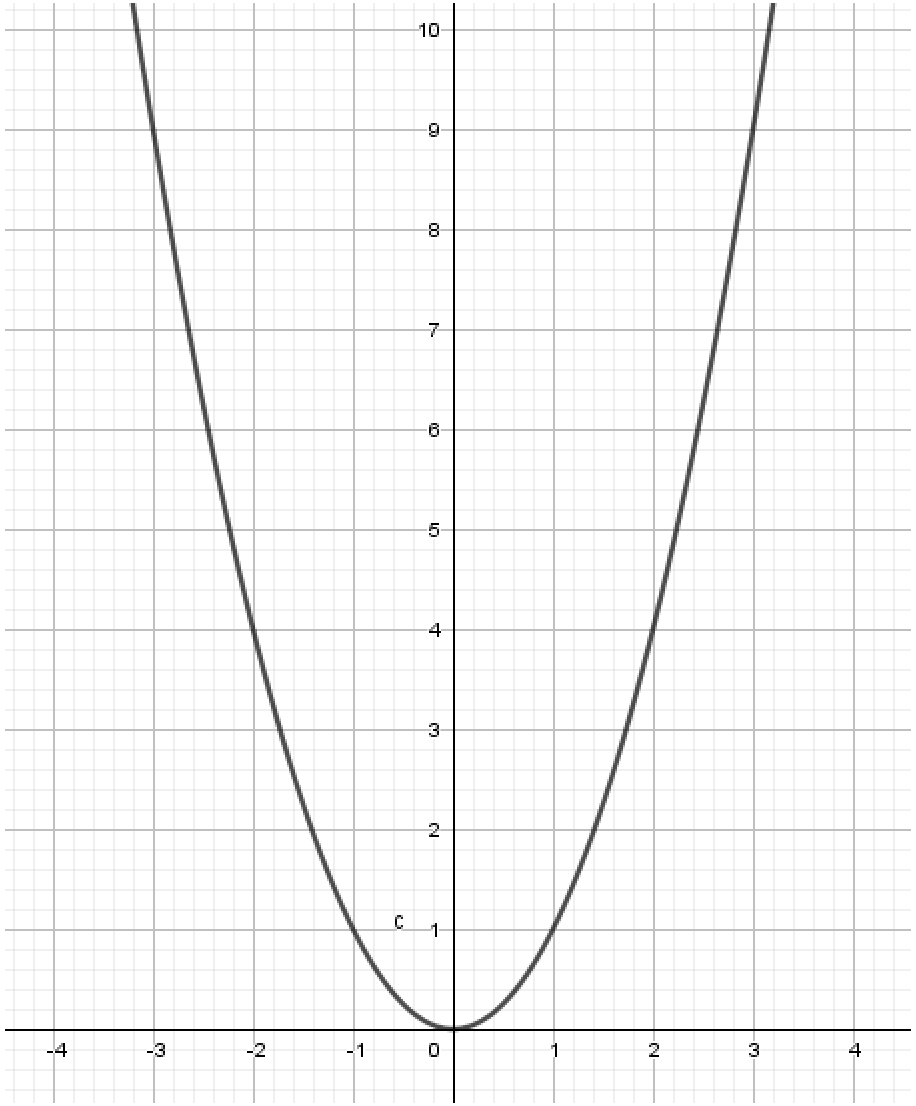
The domain of the composition fog is the set of all x in the domain of g such that $g(x)$ lies in the domain of f.

Strategy: Finding the domain and range by using a Scanner

How To Use it:	When To Use it:	Why this works?
To find the domain: 1) Start the "scanning" process from the furthest left of the relation where you see points appear going towards the right. 2) If you notice that there is an arrow on the left, this means that the points go all the way to $-\infty$. Similarly if there is an arrow pointing towards the right it will go all the way to ∞. 3) If you notice there is a break in the domain (holes, vertical asymptotes, large gaps where there are no points) keep track of where the "scanner" will not pick up points. 4) Put all of the information together using intervals where $(,)$ are used for "not including endpoints", $[,]$ are used for "including endpoints", and $\cup$ is used to join to intervals. How to find the Range: The same way as finding the domain except you scan bottom to top instead.	Finding the domain and range given a graph.	This is simply using a systematic way of finding domain and range based off of the definition.

Finding Domain and Range Examples

Example 2:
Find the domain and range of $f(x) = x^2$



Solution:
We first sketch the function to help us identify the domain and range:

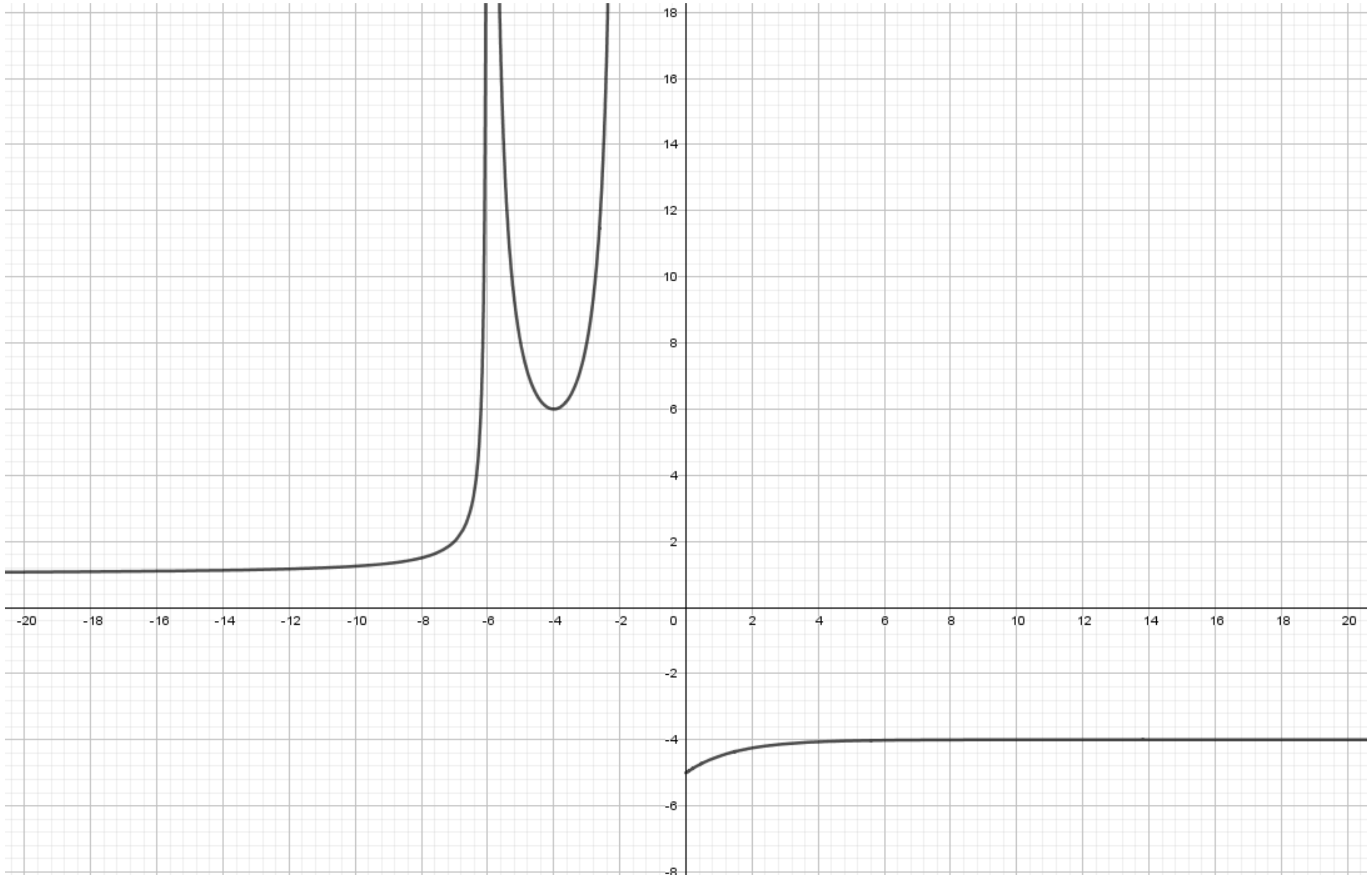
If we scan left to right, we see that we go off to $-\infty$ on the left and ∞ on the right with all numbers in between. This gives us $D = (-\infty, \infty)$

If we scan bottom to top, we see that the scanner will only pick up points starting at $y = 0$. If we continue to scan upwards, we get points all the way through ∞, thus we get $R = [0, \infty)$

Finding Domain and Range Examples

Example 3:

Given the graph below, determine the domain and range.



Solution:

$D = (-\infty, -6) \cup (-6, -2) \cup [0, \infty)$

$R = [-5, -4) \cup (1, \infty)$

Strategy: Finding the domain from an equation

How To Use it:	When To Use it:	Why this works?
To find the domain: 1) Determine different issues that appear in placing numbers into the function. You should be looking for: i) Division by zero ii) Square roots of negatives iii) Logs of 0 or negatives iv) Trig functions that are undefined (Tan, Sec, Csc and Cot) 2) Once you found all of the "problem points", create an interval that excludes all of the problem areas. It helps to construct a number line to organize the information. Note that finding the range from the equation is substantially more challenging. Typically, it is better to graph to find the range and this method only works for domain.	Finding the domain given a function.	The definition of the domain of a function is "all allowable inputs". Thus we are simply using the definition of a function in a different way.

Finding Domain Examples

Example 4:

Find the domain of $f(x) = \log|x| + \frac{\sqrt{-2x+6}}{x^2-25}$

Solution:

1) We note that we cannot have log of negatives or 0, but since $|x|$ changes all negatives to positive, we are left with only one point in the restriction of $x \neq 0$
2) We cannot divide by zero, so $x^2 - 25 = (x-5)(x+5)$ gives us two points in the restriction of $x \neq 5, -5$
3) We cannot square root negative values, thus we must have $-2x + 6 \geq 0$ and solving this gives us:

$$-2x \geq -6$$
$$x \leq 3$$

If we combine all of these together, we must have $x \leq 3$ and we are not allowed to have $x = -5{,}0{,}5$

```
−∞                 -5                     0          3          5             ∞
|--------------------|------------------------|------------|-----------|---------------|
                  Exclude                  Exclude
```

$\therefore D = (-\infty, -5) \cup (-5,0) \cup (0,3]$

Finding Domain Examples

Example 5:

Find the domain of $f(x) = \sqrt{x^6 - 1}$

Solution:

We note that we cannot have square roots of negatives, so we must have $x^6 - 1 \geq 0$

We solve this by factoring and then constructing an interval table:

$$x^6 - 1 \geq 0$$
$$(x^3 - 1)(x^3 + 1) \geq 0$$
$$(x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1) \geq 0$$

We note we cannot factor further (the discriminants are negative) so our zeros are $x = -1,1$.

We create an interval table to get:

$-\infty$ $\quad$ -1 $\quad$ 1 $\quad$ ∞

Factors	-2	0	2
$x - 1$	$-$	$-$	$+$
$x^2 + x + 1$	$+$	$+$	$+$
$x + 1$	$-$	$+$	$+$
$x^2 - x + 1$	$+$	$+$	$+$
Product	$+$	$-$	$+$

Thus we have our expression ≥ 0 on the interval $(-\infty, 1] \cup [1, \infty)$

$\therefore D = (-\infty, -1] \cup [1, \infty)$

Finding Domain

Find the domain of the following functions.

a) $f(x) = x^2 - 3$

ANS: $D = (-\infty, \infty) \quad R = [-3, \infty)$

b) $g(x) = 2 - \sqrt{x+1}$
ANS: The function is defined when $x + 1 \geq 0 \quad x \geq -1.$ So $D = [-1, \infty)$
Since range of $\sqrt{x+1} \geq 0 \Rightarrow -\sqrt{x+1} \leq 0 \Rightarrow 2 - \sqrt{x+1} \leq 2$ .So, $R = (-\infty, 2]$

c) $i(x) = \sqrt{2-x}$
ANS: The function is defined when $2 - x \geq 0 \quad x \leq 2$ .So $D = (-\infty, 2]$ and $R = [0, \infty)$

d) $j(x) = \frac{1}{x}$
ANS: The function is defined when $x \neq 0$
So $D = (-\infty, 0) \cup (0, \infty) = R$
e) $k(x) = \frac{8x}{5x-4}$
ANS: The function is defined when $x \neq \frac{4}{5}$
So $D = \left(-\infty, \frac{4}{5}\right) \cup \left(\frac{4}{5}, \infty\right)$

Finding Domain

f) Domain of $f(g(x))$

ANS: Domain of $f(g(x))$ is all x in the domain of $g(x)$ s.t. $g(x)$ is in the domain of $f(x)$

i.e. all $x\epsilon[-1,\infty)$ s.t. $2-\sqrt{x+1}\epsilon(-\infty,\infty)$.

Since $2-\sqrt{x+1}$ is always ≤ 2, the above condition holds always.

So the domain of $f(g(x)is\ [-1,\infty)$

Check: $f\left(g(x)\right)=(2-\sqrt{x=1)}^{2}-3=x-4\sqrt{x+1}+2$

Domain of $g(f(x))$

ANS: Domain of $g(f(x))$ is all x in the domain of $f(x)$ s.t. $f(x)$ is in the domain of $g(x)$

i.e. all $x\epsilon(-\infty,\infty)$ s.t. $x^2-3\geq -1 \Rightarrow x^2\geq 2 \Rightarrow x\geq\sqrt{2}\ or\ x\leq -\sqrt{2}$.

So the domain of $g(f(x)is\ \left(-\infty,-\sqrt{2}\right)\ or\ (\sqrt{2},\infty)$

Check: $g(f(x))=2-\sqrt{x^2-3+1}=2-\sqrt{x^2-2}$

Finding Domain

g) Domain of $i(j(x))$

ANS: Domain of $i(j(x))$ is all x in the domain of $j(x)$ s.t. $j(x)$ is in the domain of $i(x)$

i.e. all $x \neq 0 \; s.t. \frac{1}{x} \leq 2$

RIGHT WAY OF SOLVING THE ABOVE INEQUALITY: ***WRONG WAY OF SOLVING THE ABOVE INEQUALITY:***

$2 - \frac{1}{x} \geq 0$ $\quad \frac{2x-1}{x} \geq 0$ $\quad \boldsymbol{\frac{1}{x} \leq 2 \Rightarrow x \geq \frac{1}{2}}$

The zeros are: $x = \frac{1}{2}$ for the numerator and $x = 0$ for the denominator:

$-\infty$ $\quad$ $\mathbf{0}$ $\quad$ $\boldsymbol{\frac{1}{2}}$ $\quad$ ∞

Factors	-1	$\frac{1}{3}$	1
$2x-1$	$-$	$-$	$+$
$\frac{1}{x}$	$-$	$+$	$+$
Product	$+$	$-$	$+$

Thus we have our expression ≥ 0 on the interval $(-\infty, 0) \cup \left[\frac{1}{2}, \infty\right)$ We include $\frac{1}{2}$ as that is a zero that appear in the numerator and is defined. We exclude 0 as it is a zero that appears in the denominator and is undefined.

CHECK: $i(j(x)) = \sqrt{2 - \frac{1}{x}}$

Finding Domain

Domain of $j(i(x))$

ANS: Domain of $j(i(x))$ is all x in the domain of $i(x)$ s.t. $i(x)$ is in the domain of $j(x)$

i.e. all $x \leq 2$ s.t. $\sqrt{2-x} \neq 0 \Rightarrow 2-x > 0 \quad x < 2$

So, domain is $= (-\infty, 2)$

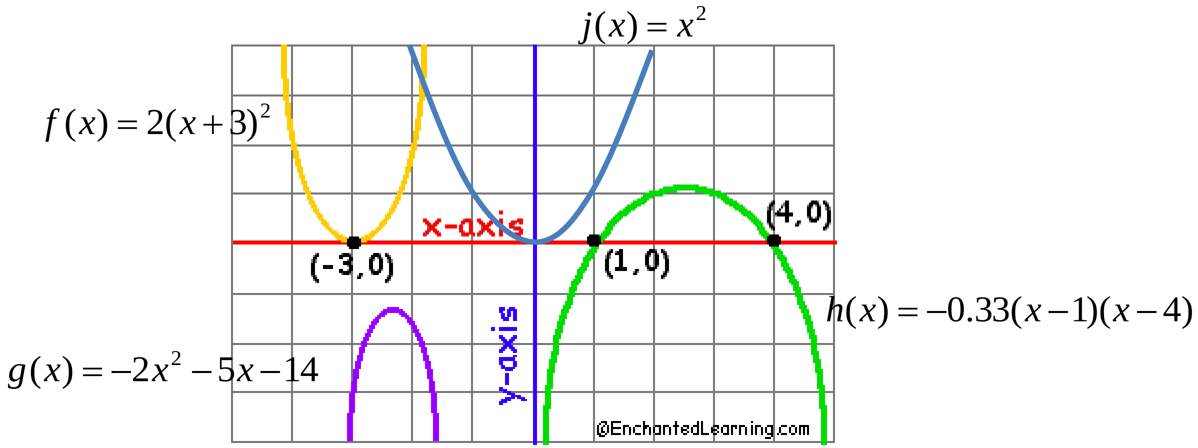
Check: $j\big(i(x)\big) = \frac{1}{\sqrt{2-x}}$

Definition: Parent Function

A Parent Function is a function from a family of functions with the simplest equation that still maintains the shape.

Example 1:

Given the family of quadratic functions below, which would be the parent function?



Solution:

$j(x) = x^2$ would be the parent function

Strategy: Graphing Parent Functions using a Table of Values

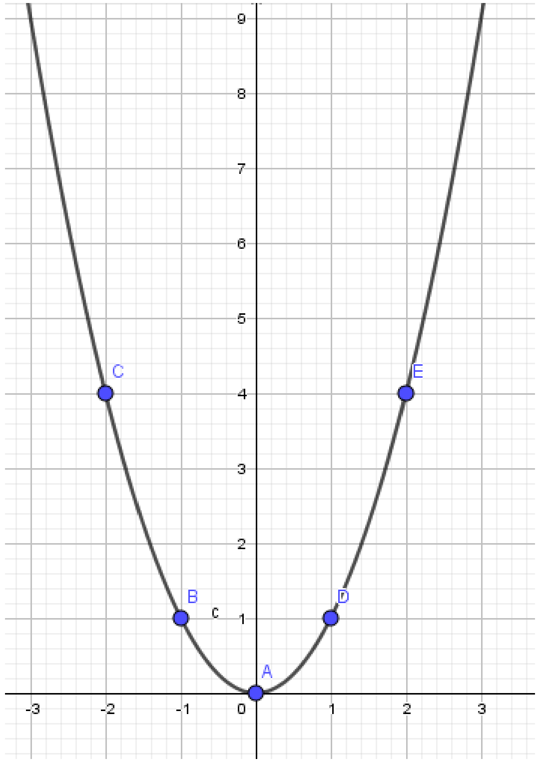
How To Use it:	**When To Use it:**	**Why this works?**
Start with a set of x – values (think of ones that will be easy inputs for the function) and use the function to generate corresponding y – values. You can then plot the points on the grid.	Although it can be used to graph any function, it is best when graphing parent functions (as it is possible to miss key information).	This is simply a way to organize your function to set up a graph.

Example 1:

Using a table of values, come up with the graph for $f(x) = x^2$

Solution:

x	$f(x)$
-2	$(-2)^2 = 4$
-1	$(-1)^2 = 1$
0	$(0)^2 = 0$
1	$1^2 = 1$
2	$2^2 = 4$



Examples: Graphing Parent Functions

Example 2:

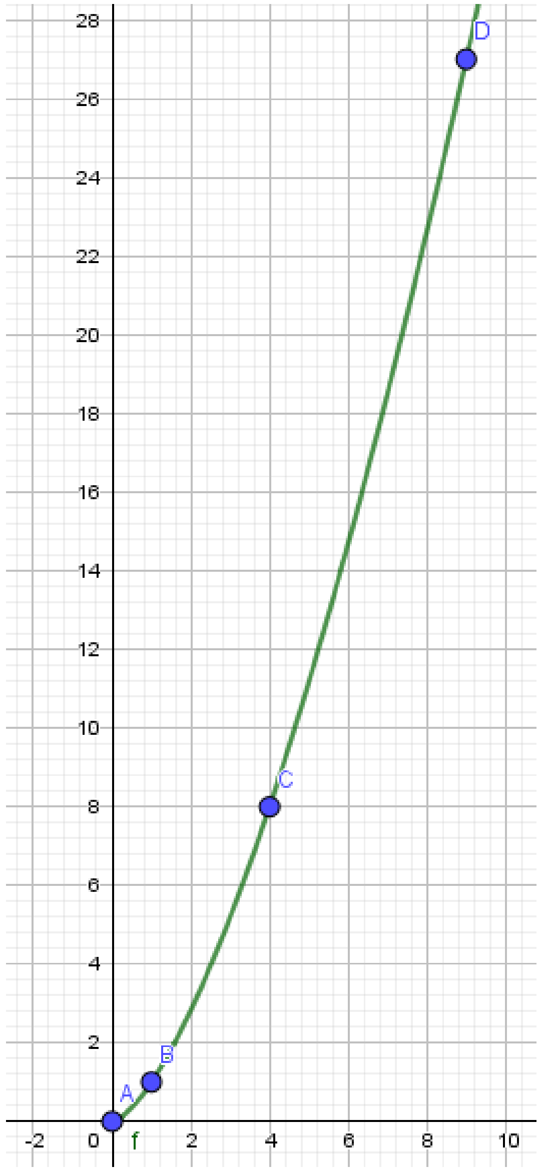
Using a table of values, come up with the graph for $g(x) = x^{3/2}$

Solution:

Using our exponent laws, we see that $g(x) = x^{3/2} = \sqrt{x}^3$

Thus we choose the following inputs as they are easy to square root:

x	$f(x)$
0	$\sqrt{0}^3 = 0$
1	$\sqrt{1}^3 = 1$
4	$\sqrt{4}^3 = 8$
9	$\sqrt{9}^3 = 27$



Definition: Piecewise Functions

A piecewise function is a function made up of multiple other functions where each “piece” is defined on its own domain.

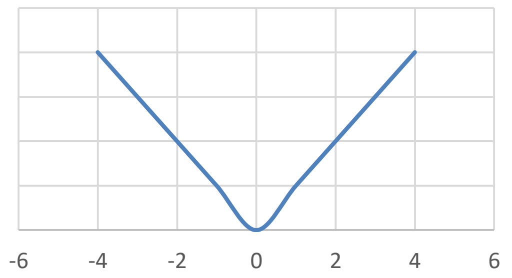
Example of Piecewise notation with 3 pieces: $f(x) = \begin{cases} h(x) & x \in Domain_1 \\ i(x) & x \in Domain_2 \\ j(x) & x \in Domain_3 \end{cases}$

Since functions can end and start abruptly, we recall that we use closed circles to indicate that we reach that point ( ) and open circles to indicate we reach close to that point, but never reach that actual point ( ).

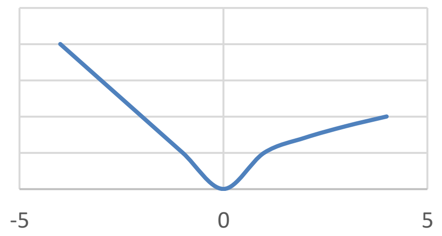
Piecewise defined Function or Split function

Draw the graph of the split function

a) $f(x) = \begin{cases} x \ if \ x \geq 0 \\ -x \ if \ x < 0 \end{cases}$



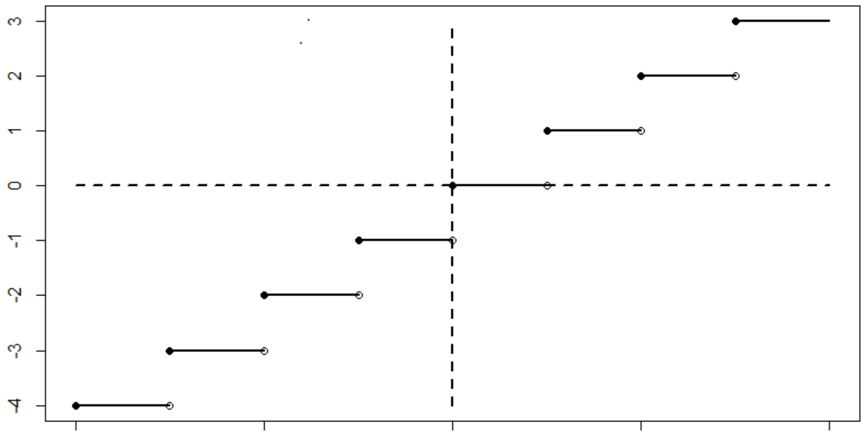
b) $f(x) = \begin{cases} \sqrt{x} \ if \ x \geq 0 \\ -x \ if \ x < 0 \end{cases}$



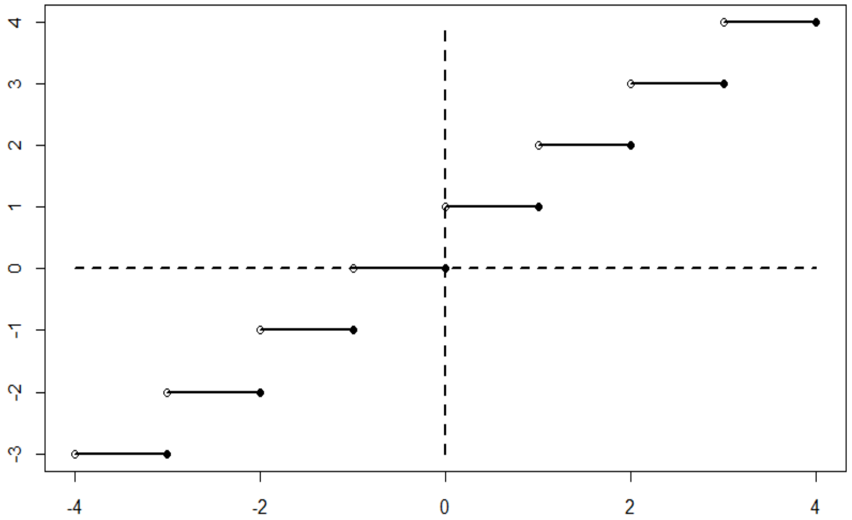
Piecewise defined Function or Split function

c) The Greatest Integer function $f(x) = \lfloor x \rfloor$

whose value is the greatest integer $\leq$ to x.



d) The Least Integer function $f(x) = \lceil x \rceil$
whose value is the smallest integer $\geq$ to x.



Examples: Piecewise Functions

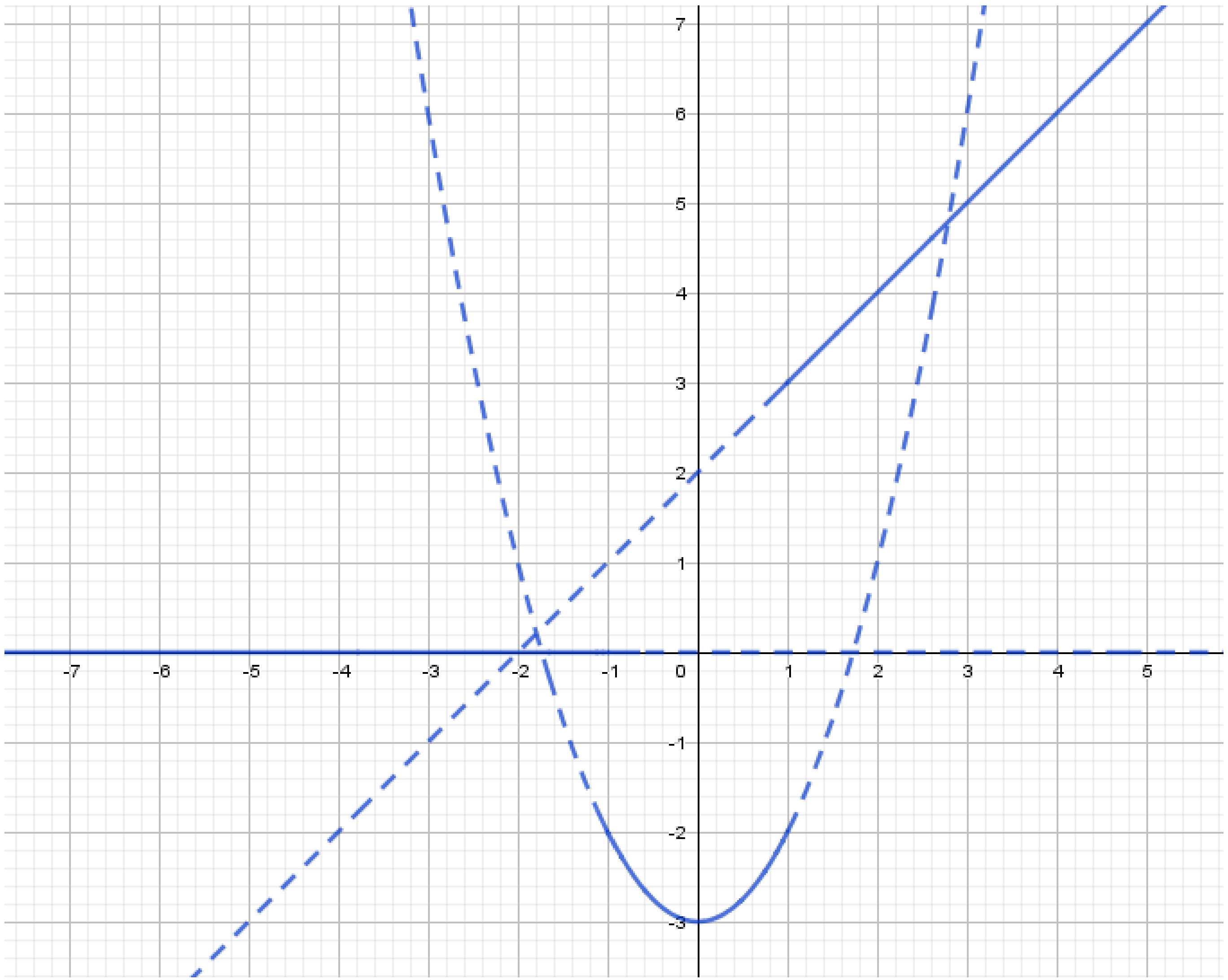
Example 1:

Graph the following piecewise function:

$$f(x) = \begin{cases} x + 2 & x > 1 \\ x^2 - 3 & -1 \leq x \leq 1 \\ 0 & x < -1 \end{cases}$$

Solution:

To graph the function, it helps to graph each function separately, then erase the part of the domain that is unneeded.



Examples: Piecewise Functions

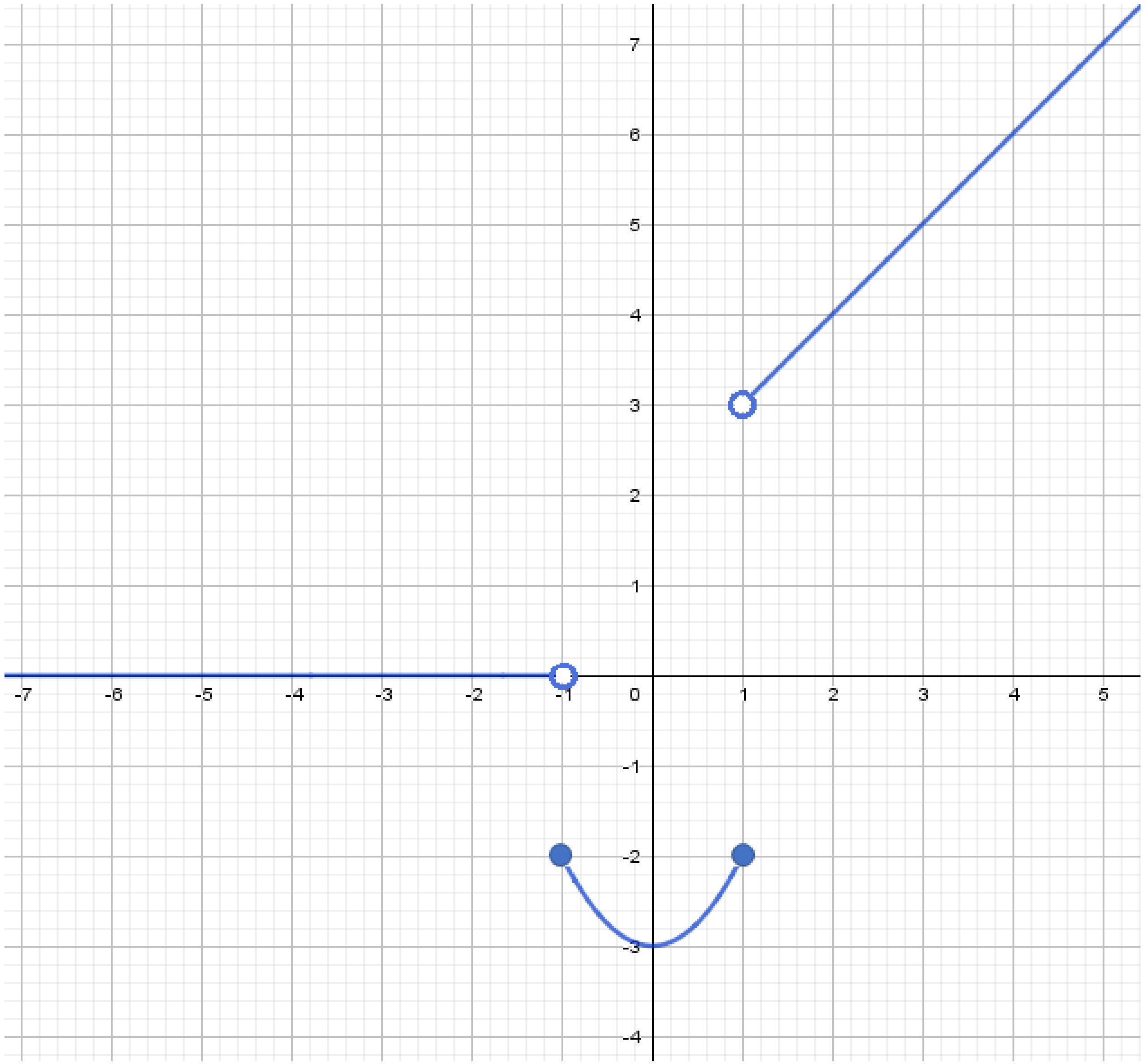
Example 1 Continued:

Graph the following piecewise function:

$$f(x) = \begin{cases} x+2 & x > 1 \\ x^2 - 3 & -1 \le x \le 1 \\ 0 & x < -1 \end{cases}$$

Solution:

Finally we make sure to plot the correct endpoints.



Strategy: Removing Absolute Values From Functions

How To Use it:	When To Use it:	Why this works?
1) Use the fact that: $$\lvert f(x)\rvert = \begin{cases} f(x) & f(x) \geq 0 \\ -f(x) & f(x) < 0 \end{cases}$$ 2) Isolate the x in the domains (note that this may require you to solve a polynomial inequality).	When you want to convert absolute value functions in to piecewise functions (this will help with limits that will be seen in the future).	The definition of an absolute value function is $\lvert\square\rvert$ is $\square$ when the value is positive and (since multiplying negatives by -1 gives a positive) $-\square$ when $\square$ is negative.

Example 1:
Convert the following to a piecewise function: $f(x) = \lvert 6 + 2x\rvert$

Solution:

$$\lvert 6 + 2x\rvert = \begin{cases} 6 + 2x & 6 + 2x \geq 0 \\ -(6 + 2x) & 6 + 2x < 0 \end{cases}$$

We then solve the inequality:

$$\begin{aligned} 6 + 2x &\geq 0 \\ 6 &\geq -2x \\ -3 &\leq x \end{aligned}$$

(dividing by negatives swaps the inequality)

Solving the other inequality is identical except all of the signs are pointing in the opposite direction. This gives us our piecewise function:

$$\lvert 6 + 2x\rvert = \begin{cases} 6 + 2x & x \geq -3 \\ -6 - 2x & x < -3 \end{cases}$$

Examples: Removing Absolute Values

Example 2:

Convert the following to a piecewise function: $f(x) = \frac{1}{|x^2+x-6|}$

Solution:

$$\frac{1}{|x^2+x-6|} = \begin{cases} \dfrac{1}{x^2+x-6} & x^2+x-6 \geq 0 \\ \dfrac{1}{-(x^2+x-6)} & x^2+x-6 < 0 \end{cases}$$

We then solve the inequality:

$x^2 + x - 6 \geq 0$

$(x+3)(x-2) \geq 0$

Thus our zeroes are $x = -3, 2$. We can now create our interval table:

$-\infty$ | -3 | 2 | ∞

Factors	-4	0	3
$x+3$	$-$	$+$	$+$
$x-2$	$-$	$-$	$+$
Product	$+$	$-$	$+$

Thus we have our expression > 0 on the interval $(-\infty, -3) \cup (2, \infty)$. The expression is < 0 on the interval $(-3,2)$

We note that we cannot have $x = -3$ or $x = 2$ as they will produce a division by zero.

$$\therefore f(x) = \begin{cases} 1/x^2+x-6 & x \in (-\infty, -3) \cup (2, \infty) \\ 1/-(x^2+x-6) & x \in (-3,2) \end{cases}$$

Strategy: Removing Absolute values from linear inequalities

How To Use it:	When To Use it:	Why this works?
If we are given b is some positive number, then: 1) If we are given $\lvert\square\rvert < b$ we can change this to the form : $-b < \square < b$ 2) If we are given $\lvert a\rvert > b$ we can change this to the form: $\square > b$ or $\square < -b$	When you want to convert absolute value inequalities to intervals.	From the definition of absolute value, we can only have $\lvert\square\rvert < b$ when the values of $\square$ are stuck between $-b$ and b. Any other number will force $\lvert\square\rvert \geq b$, which means if we want to solve this inequality, it would be the other cases of either $\square \geq b$ or $\square \leq -b$

Example 3:

Determine an interval such that $|x-2| < 7$ and $x \geq 3$.

Solution:

We first remove the absolute value in the inequality:

$$|\square| < b \rightarrow -b < \square < b$$
$$\rightarrow -7 < \square < 7$$
$$\rightarrow -7 < x-2 < 7$$
$$\rightarrow -5 < x < 9 \qquad \text{(add 2 to the inequalities to isolate } x\text{)}$$

Since we also require $x \geq 3$, it will help to draw a number line to find our interval(s):

$-\infty$	-5	3	9	∞	
|--------------------|	--------------------|	--------------------|	--------------------|		$-5 < x < 9$
|--------------------|	--------------------|	--------------------|	--------------------|		$x \geq 3$
	Exclude	Include	Exclude		

Since we need to be in both intervals, this gives us that $x \in [3,9)$

Increasing and Decreasing Function

A function f is called (strictly) increasing on an interval I if $f(a) < f(b)$ whenever $a < b$ in I.

A function f is called (strictly) decreasing on an interval I if $f(a) > f(b)$ whenever $a < b$ in I.

Symmetry

A function f is called even if it satisfies $f(-x) = f(x)$

Example: The function $f(x) = x^2$ is even since

$$f(-x) = (-x)^2 = x^2 = f(x)$$

A function f is called odd if it satisfies $f(-x) = -f(x)$

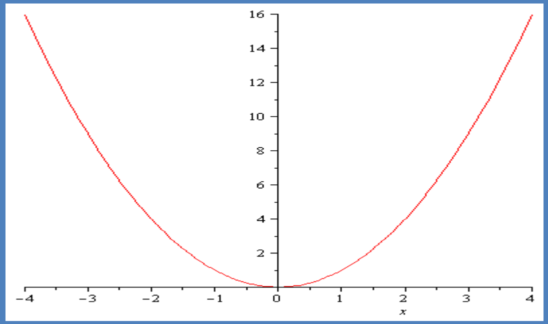
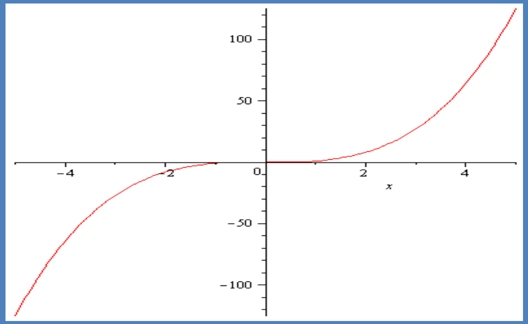
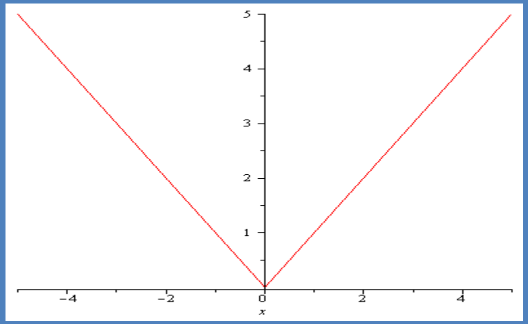
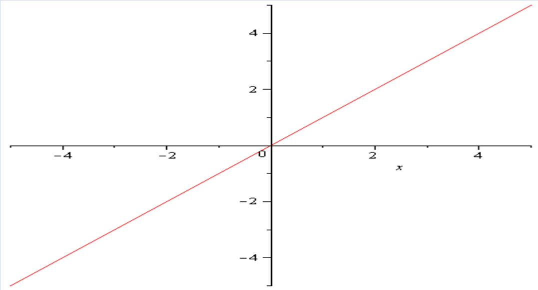
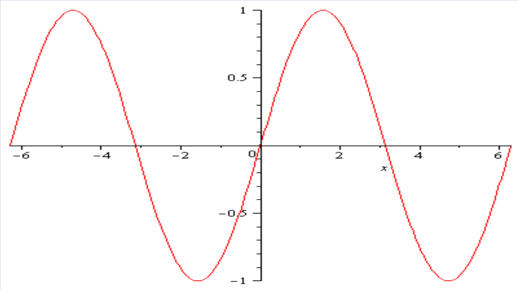
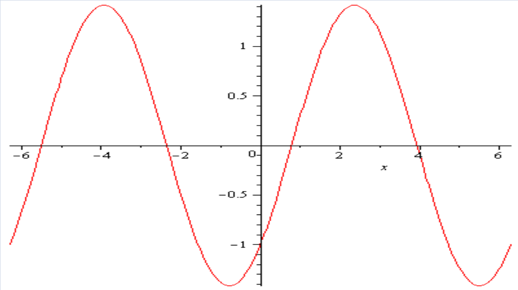
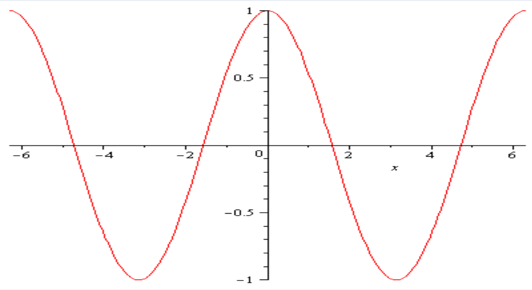
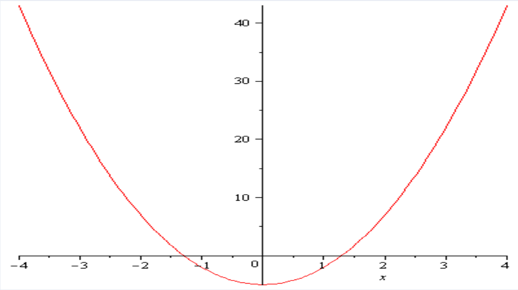
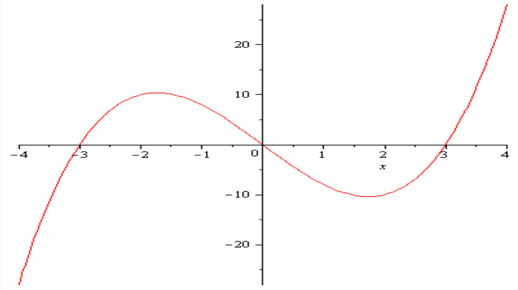
Example: The function $f(x) = x^3$ is odd since

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

Examples: Even and Odd Functions

Example:

Which of the following graphs represents even functions, odd functions, or neither even nor odd:

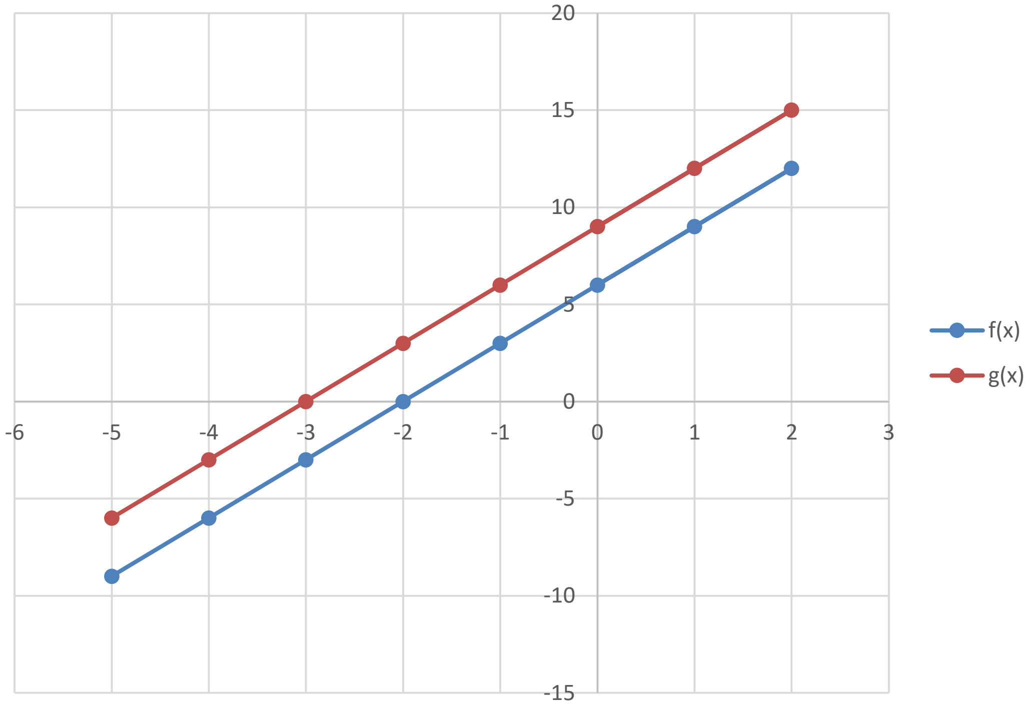
	Even		Odd		Even
	Odd		Odd		Neither
	Even		Even		Odd

Transformation of Graphs

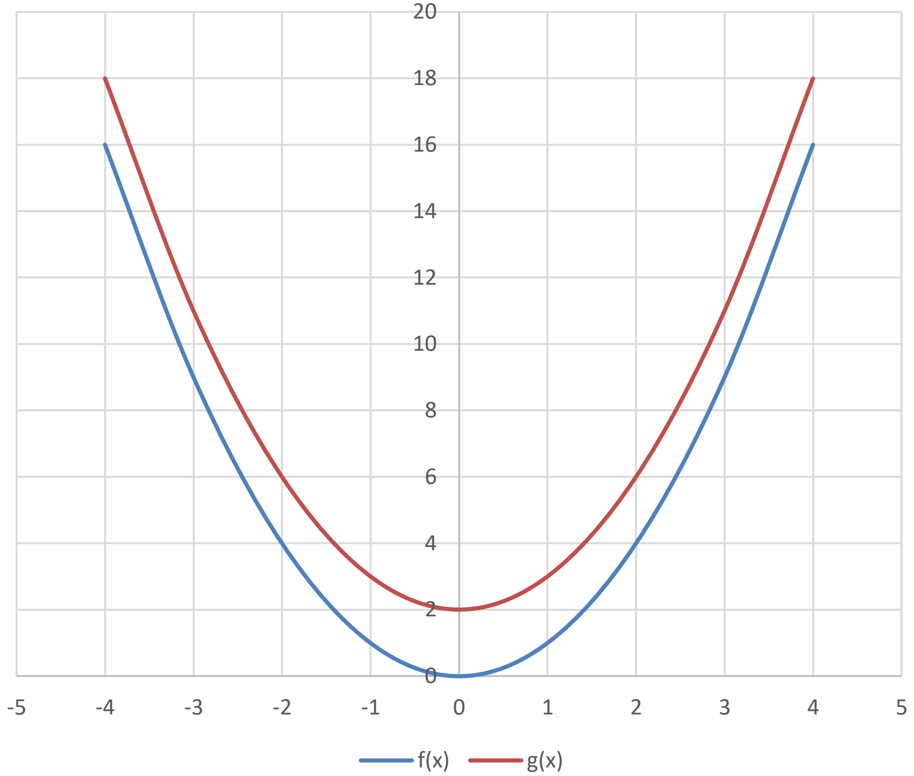
Shifting

Given a function *f(x)* and a positive constant *k*, the function g(x)=*f(x)±k* shifts the graph up/down by *k* units.

Example: $f(x) = 3x + 6$; $g(x) = (3x + 6) + 3$



Example: $f(x) = x^2$; $g(x) = f(x) + 2$

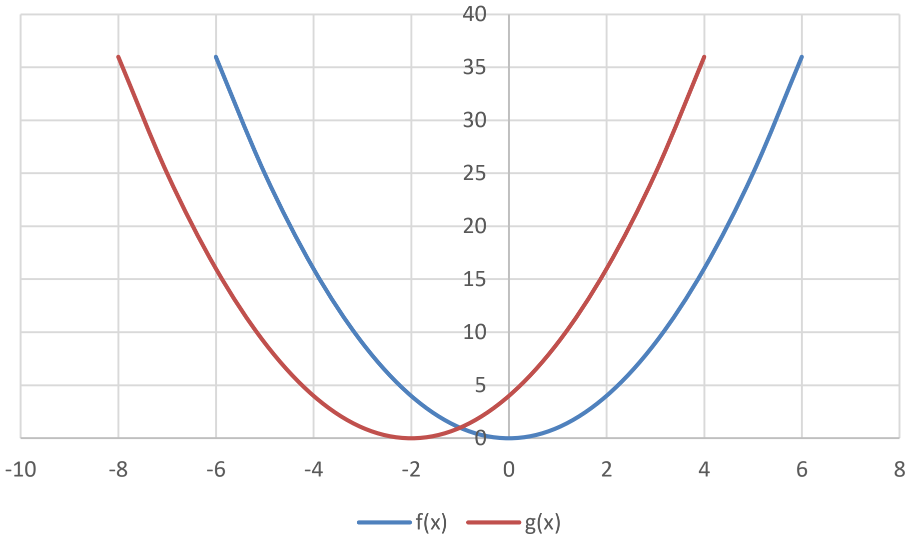


Transformation of Graphs

Shifting

Given a function *f(x)* and a positive constant *k*, the function g(x)=*f(x±h)* shifts the graph left/right by *h* units.

Example: $f(x) = x^2$; $g(x) = f(x+2) = (x+2)^2$



Transformation of Graphs

Shifting

Example: Find the equation of a function which shifts down the graph of the function $f(x) = x^2$ by 2 units.

ANS: $g(x) = x^2 - 2$

Example: Find the equation of a function which shifts up the graph of the function $f(x) = x^2 + x + 3$ by 2 units.

ANS: $g(x) = x^2 + x + 5$

Example: Find the equation of a function which shifts the graph of the function $f(x) = x^2 + x + 3$ by 2 units to the right.

ANS:$g(x) = (x-2)^2 + (x-2) + 3 = x^2 - x + 4$

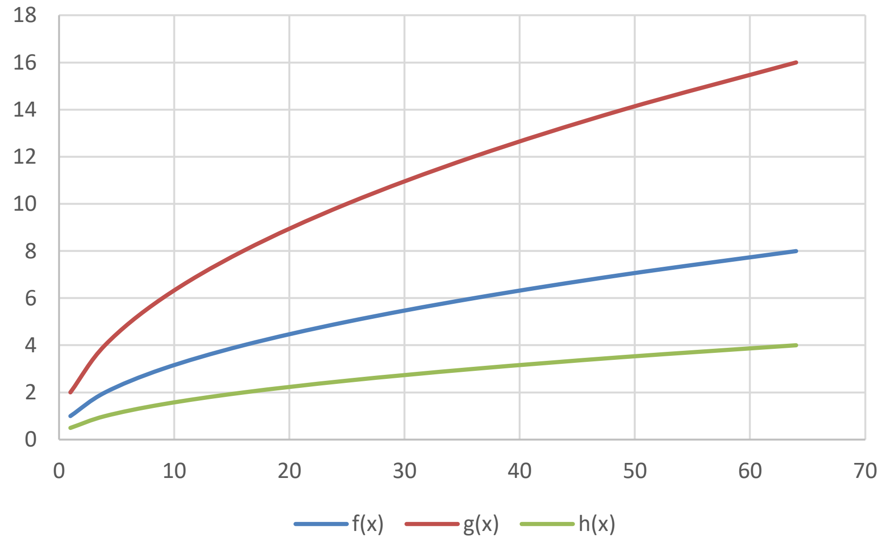
Transformation of Graph

Stretching

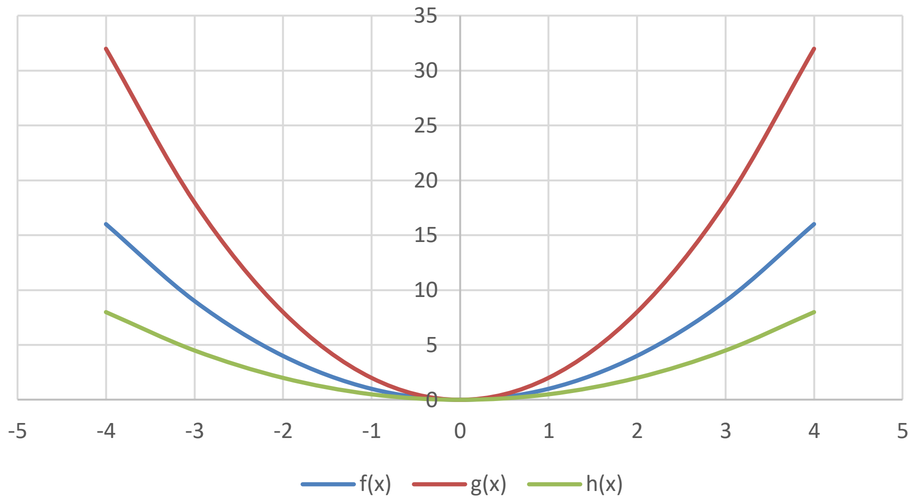
Given a function $f(x)$ and a positive constant c>1, then function $g(x) = c.f(x)$ stretches the graph vertically by a factor of c.

Given a function $f(x)$ and a positive constant c>1, then function $g(x) = \frac{1}{c}.f(x)$ compresses the graph vertically by a factor of c.

Example: $f(x) = \sqrt{x}; g(x) = 2.\sqrt{x}\ ; h(x) = \frac{1}{2}\sqrt{x}$



Example: $f(x) = x^2; g(x) = 2.x^2\ ; h(x) = \frac{1}{2}x^2$



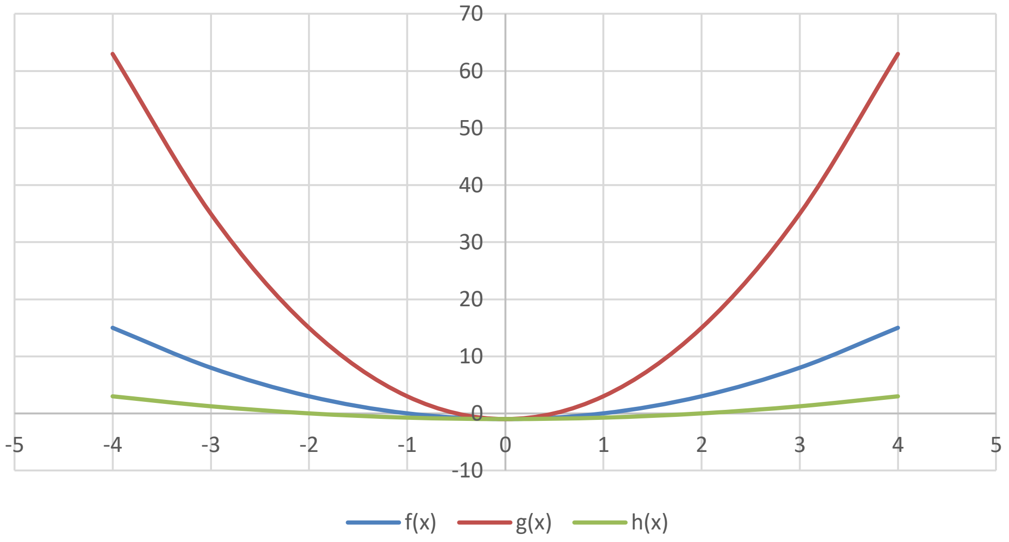
Transformation of Graph

Stretching

Given a function $f(x)$ and a positive constant c>1, , then function$g(x) = f(cx)$ compresses the graph horizontally by a factor of c.

Given a function $f(x)$ and a positive constant c>1, then function$g(x) = f(\frac{1}{c}x)$ stretches the graph horizontally by a factor of c.

Example: $f(x) = x^2 - 1$; $g(x) = (2x)^2 - 1 = 4x^2$-1 ; $h(x) = (\frac{1}{2}\ x)^2 - 1 = \frac{1}{4}x^2$-1



Transformation of Graph

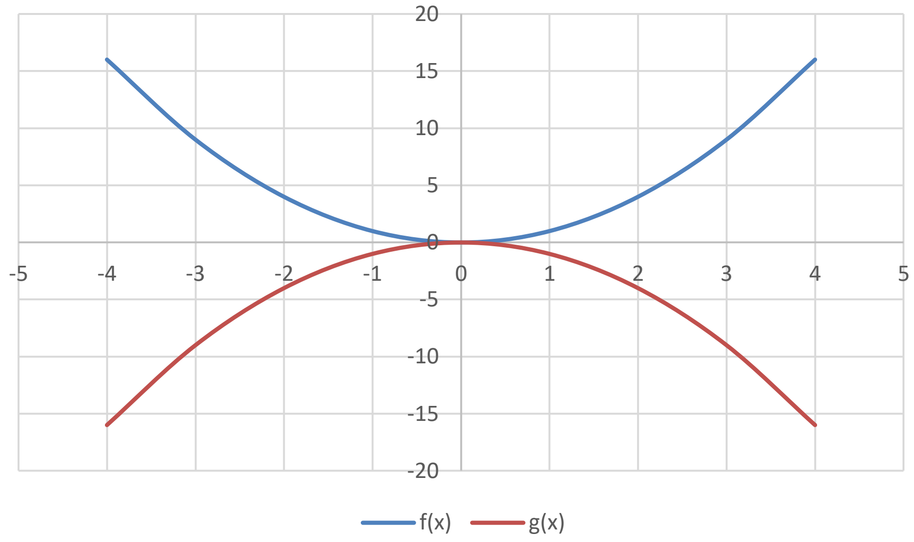
Reflection

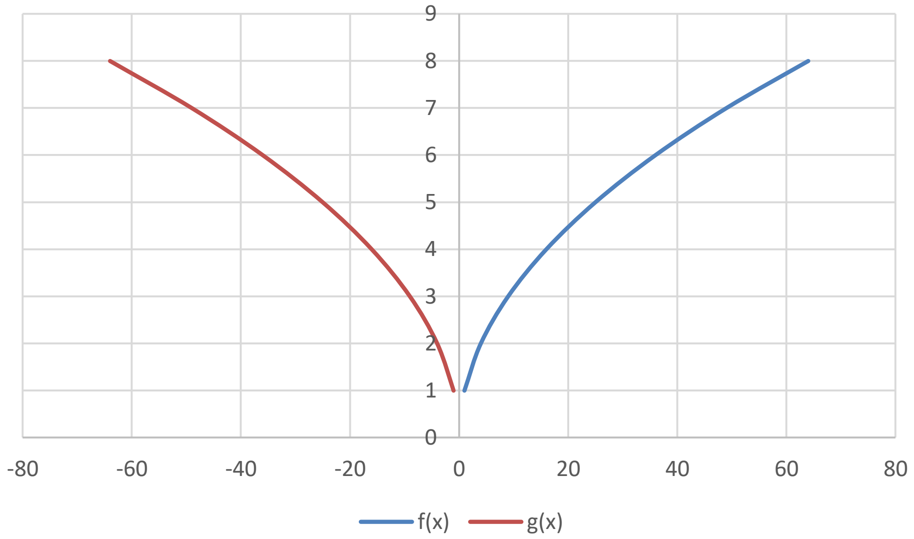
Given a function $f(x)$, then function $g(x) = -f(x)$ reflects the graph about x-axis.

Given a function $f(x)$, then function $g(x) = f(-x)$ reflects the graph about y-axis.

Example: $f(x) = x^2$; $g(x) = -x^2$;

$f(x) = \sqrt{x}$; $g(x) = \sqrt{-x}$





Strategy: Graphing Functions using Transformations

How To Use it:	When To Use it:	
1) Identify the parent function $f(x)$. 2) Write the function in the form: $g(x) = a[f(b[x+c])] + d$ 3) Construct the table of values for the parent function. 4) Apply the following transformations to the table of values to get the new table of values: New x values: $\frac{x}{b} - c$ New y values: $(y \times a) + d$ 5) Graph the new function.	When you have a function that is not a parent function and you wish to graph this function.	

Examples: Graphing With Transformations

Example 1:

Graph the following equation: $g(x) = 3x^2 - 2$

Solution:

Parent function $f(x) = x^2$

$g(x) = 3[f(1[x + 0])] - 2$

$a = 3, b = 1, c = 0, d = -2$

We generate our table of values for the parent function:

x	$f(x)$
-2	$(-2)^2 = 4$
-1	$(-1)^2 = 1$
0	$(0)^2 = 0$
1	$1^2 = 1$
2	$2^2 = 4$

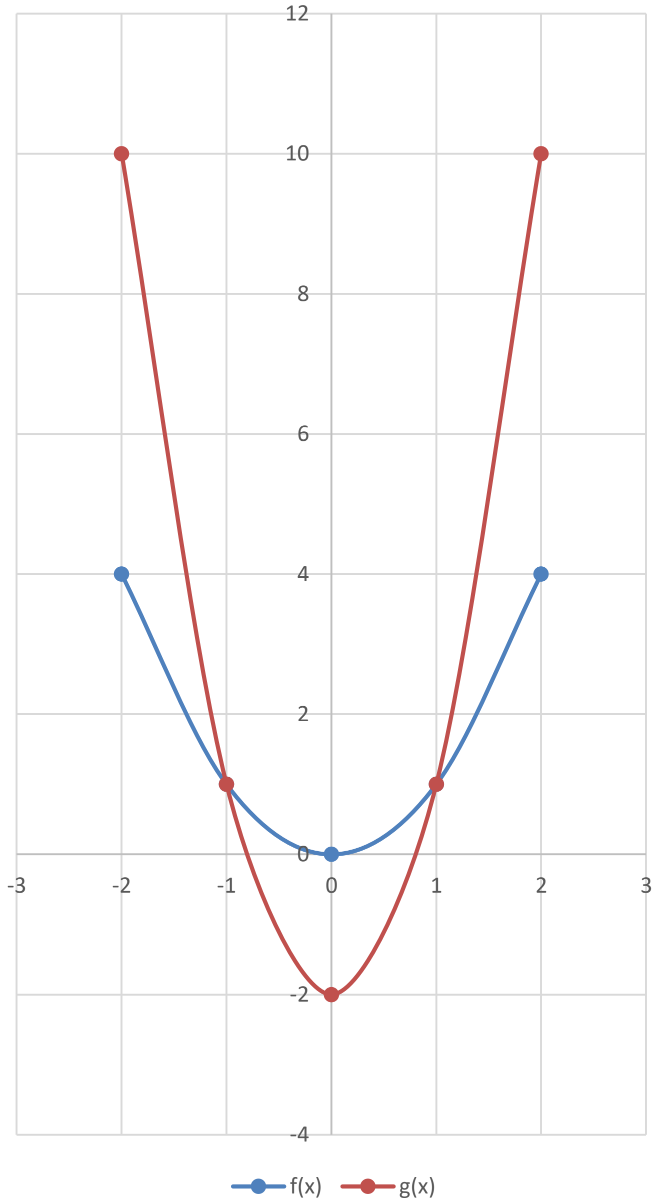
New x values:

$\frac{x}{b} - c \rightarrow \frac{x}{1} - 0$

New y values:

$(y \times a) + d \rightarrow 3y - 2$

New x	$g(x)$
$\frac{-2}{1} - 0 = -2$	$3(4) - 2 = 10$
$\frac{-1}{1} - 0 = -1$	$3(1) - 2 = 1$
$\frac{0}{1} - 0 = 0$	$3(0) - 2 = -2$
$\frac{1}{1} - 0 = 1$	$3(1) - 2 = 1$
$\frac{2}{1} - 0 = 2$	$3(4) - 2 = 10$



Examples: Graphing With Transformations

(self reading)

Example 2:

Graph the following equation: $h(x) = -\left(2^{2(x-1)}\right) - 2$

Solution:

Parent function $f(x) = 2^x$

$h(x) = -1[f(2[x-1])] - 2$

$a = -1, b = 2, c = -1, d = -2$

We generate our table of values for the parent function:

x	$f(x)$
-1	$2^{-1} = 1/2$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$

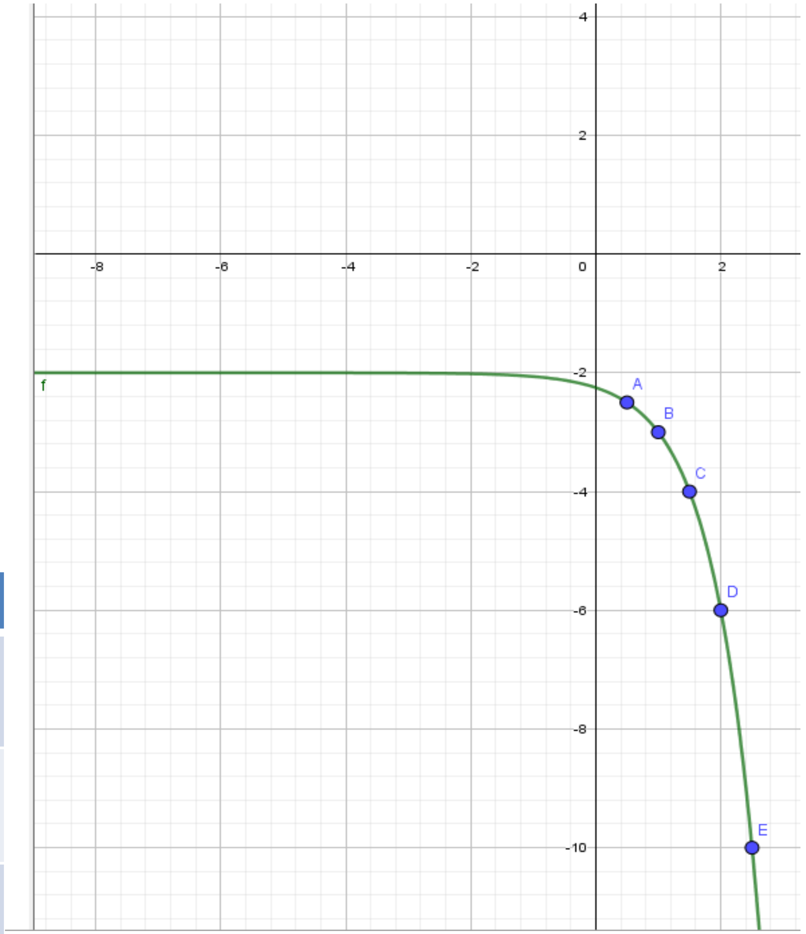
New x values:

$\frac{x}{b} - c \rightarrow \frac{x}{2} - (-1)$

New y values:

$(y \times a) + d \rightarrow -y - 2$

New x	$\mathbf{h}(x)$
$\frac{-1}{2} + 1 = 0.5$	$-\frac{1}{2} - 2 = -2.5$
$\frac{0}{2} + 1 = 1$	$-(1) - 2 = -3$
$\frac{1}{2} + 1 = 1.5$	$-(2) - 2 = -4$
$\frac{2}{2} + 1 = 2$	$-(4) - 2 = -6$
$\frac{3}{2} + 1 = 2.5$	$-(8) - 2 = -10$



Note that we can apply this process to the asymptote as well. Since the asymptote of $y = 2^x$ is $y = 0$, we get that the new asymptote will occur at $y = -1(0) - 2 = -2$.

Examples: Graphing With Transformations

(self reading)

Example 3:

Graph the following equation: $k(x) = \sqrt{-6 - 3x} + 4$

Solution:

Parent function $f(x) = \sqrt{x}$

We rewrite the function as

$$k(x) = \sqrt{-3[x+2]} + 4$$

$$k(x) = 1[f(-3[x+2])] + 4$$

$$a = 1, b = -3, c = 2, d = 4$$

We generate our table of values for the parent function:

x	$f(x)$
0	$\sqrt{0} = 0$
1	$\sqrt{1} = 1$
4	$\sqrt{4} = 2$
9	$\sqrt{9} = 3$
16	$\sqrt{16} = 4$

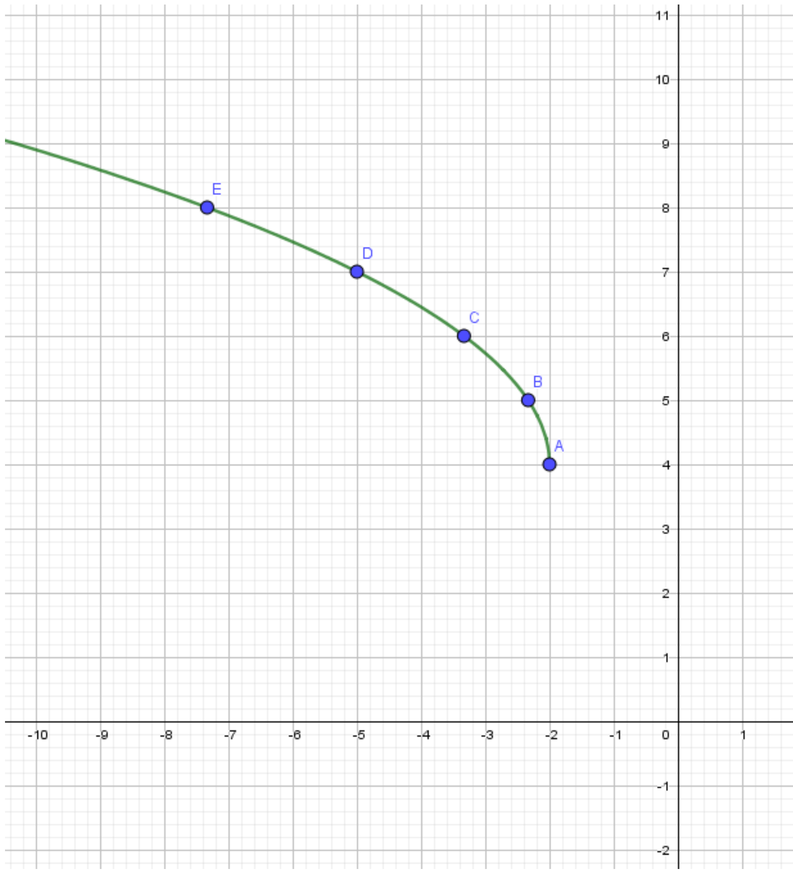
New x values:

$$\frac{x}{b} - c \rightarrow \frac{x}{-3} - (2)$$

New y values:

$$(y \times a) + d \rightarrow y + 4$$

New x	$\mathbf{k}(x)$
$\frac{0}{-3} - (2) = -2$	$0 + 4 = 4$
$\frac{1}{-3} - (2) = -2.3$	$1 + 4 = 5$
$\frac{4}{-3} - (2) = -3.3$	$2 + 4 = 6$
$\frac{9}{-3} - (2) = -5$	$3 + 4 = 7$
$\frac{16}{-3} - (2) = -7.3$	$4 + 4 = 8$



Examples: Graphing With Transformations

Example 4:

Given $f(x)$ as the graph to the right, determine $g(x) = f(2x - 6)$

Solution:

Parent function is given in the graph

We rewrite the function as

$g(x) = f(2[x - 3])$

$a = 1, b = 2, c = -3, d = 0$

For our table of values, we record key points from our graph:

x	$f(x)$
0	0
3	6
5	6
8	3

x	$f(x)$
0	0
3	6
5	6
8	3

New x values:

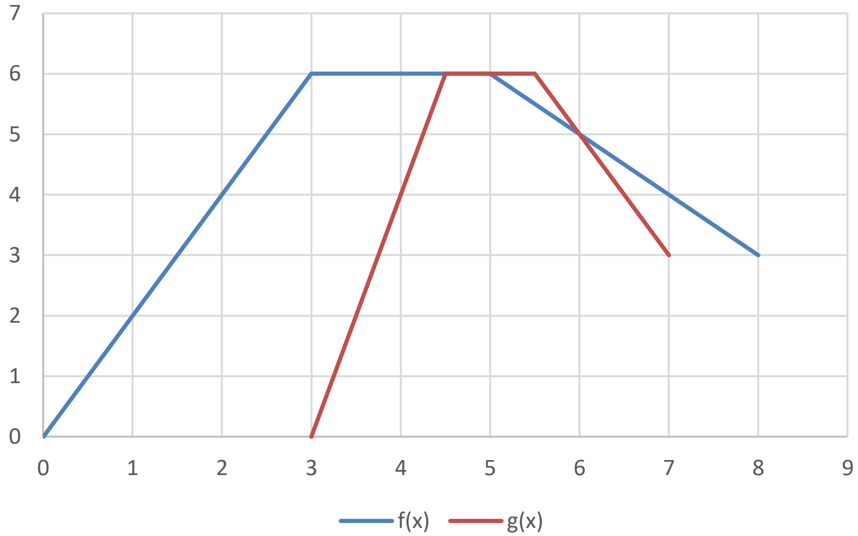
$\frac{x}{b} - c \rightarrow \frac{x}{2} - (-3)$

New y values:

$(y \times a) + d \rightarrow y$

(no change)

New x	g(x)
$\frac{0}{2} - (-3) = 3$	0
$\frac{3}{2} - (-3) = 4.5$	6
$\frac{5}{2} - (-3) = 5.5$	6
$\frac{8}{2} - (-3) = 7$	3



Examples: Graphing With Transformations

Example 4(cont):

Given $f(x)$ as the graph to the right, determine $g(x) = 1 - f(x)$

Solution:

Parent function is given in the graph

We rewrite the function as

$g(x) = (-1)f(1[x - 0])+1$

$a = -1, b = 1, c = 0, d = 1$

For our table of values, we record key points from our graph:

x	$f(x)$
0	0
3	6
5	6
8	3

New x values:

$\frac{x}{b} - c \rightarrow x$

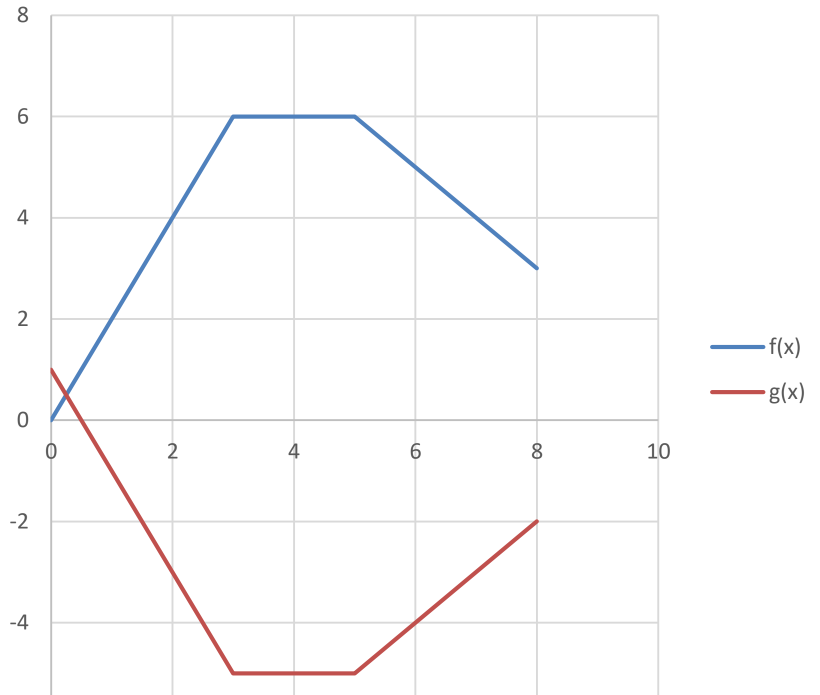
(no change)

New y values:

$(y \times a) + d \rightarrow -y + 1$

New x	$g(x)$
0	1
3	-5
5	-5
8	-2

New x	$g(x)$
0	1
3	-5
5	-5
8	-2



Examples: Piecewise Functions

Example 5:

Determine the equation of the graph given to the right:

Solution:

First Piece: We see that the first function is an exponential function with base of 2 (as it doubles as it goes to the right). This gives us 2^x

Second Piece: We see that at $x = 4$ it changes to a linear function. It has a slope of 1 and (if we extend the line) we see it has a y −int of 2. This gives us $x + 2$.

Third Piece: A single point at $(4,0)$

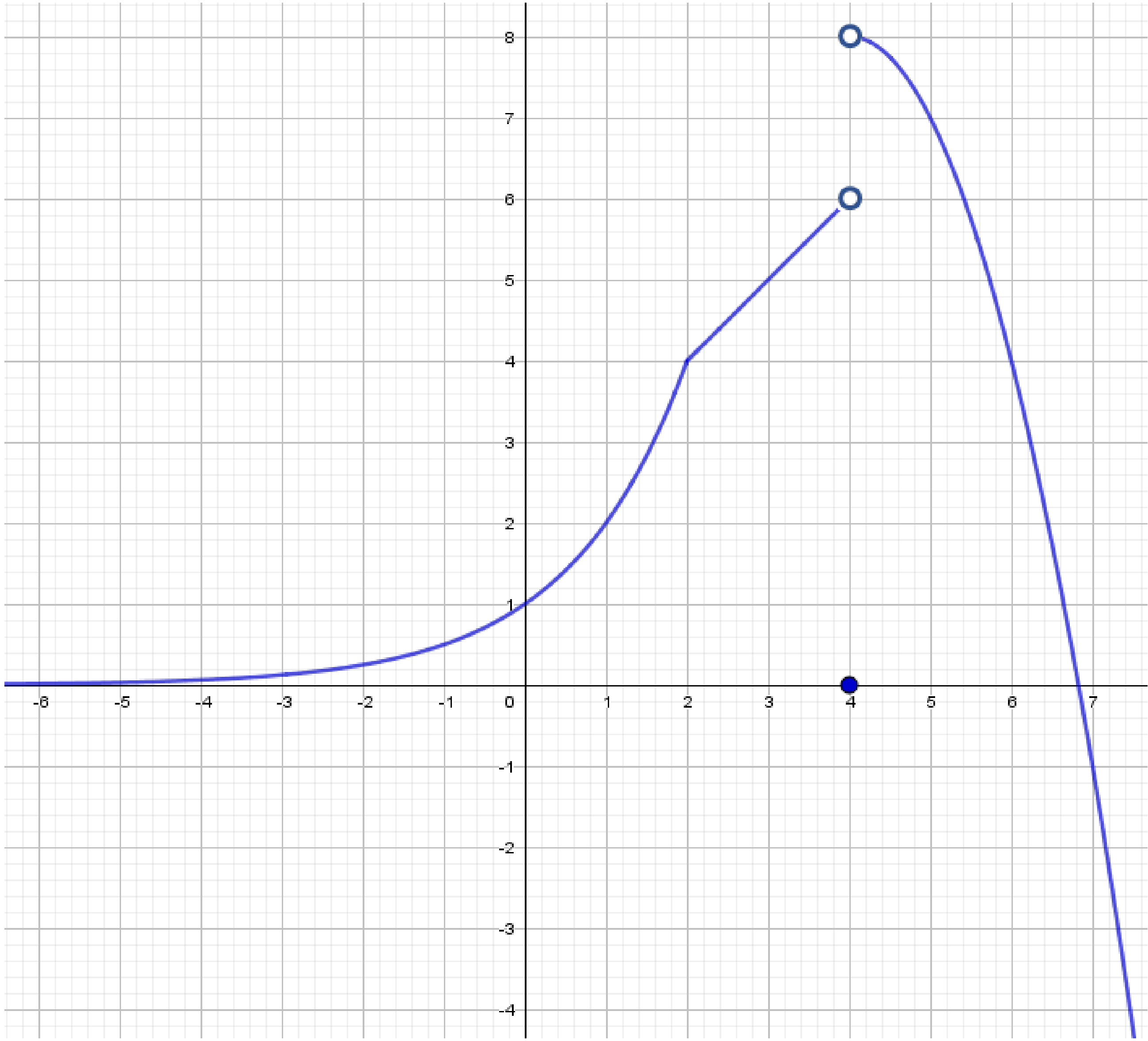
Last Piece: A negative quadratic (as it opens down) shifted left 4 and up 8. We note that we can find the a value by solving:

$7 = a(5-4)^2 + 8 \quad \rightarrow -1 = a.$

This gives us the quadratic $-(x-4)^2 + 8$

Putting all pieces together gives us our piecewise function:

$$f(x) = \begin{cases} 2^x & x \leq 2 \\ x+2 & 2 < x < 4 \\ 0 & x = 4 \\ -(x-4)^2 + 8 & x > 4 \end{cases}$$



Finding Domain and Range Examples

(self reading)

Example 6:

Determine the domain and range of the function $g(x) = 3|x-2|+1$

Solution:

Parent function $f(x) = |x|$

$g(x) = 3[f([x-2])]+1$

$a = 3, b = 1, c = -2, d = 1$

We generate our table of values for the parent function:

x	$f(x)$
-2	$\|-2\| = 2$
-1	$\|-1\| = 1$
0	$\|0\| = 0$
1	$\|1\| = 1$
2	$\|2\| = 2$

We then apply the transformations:

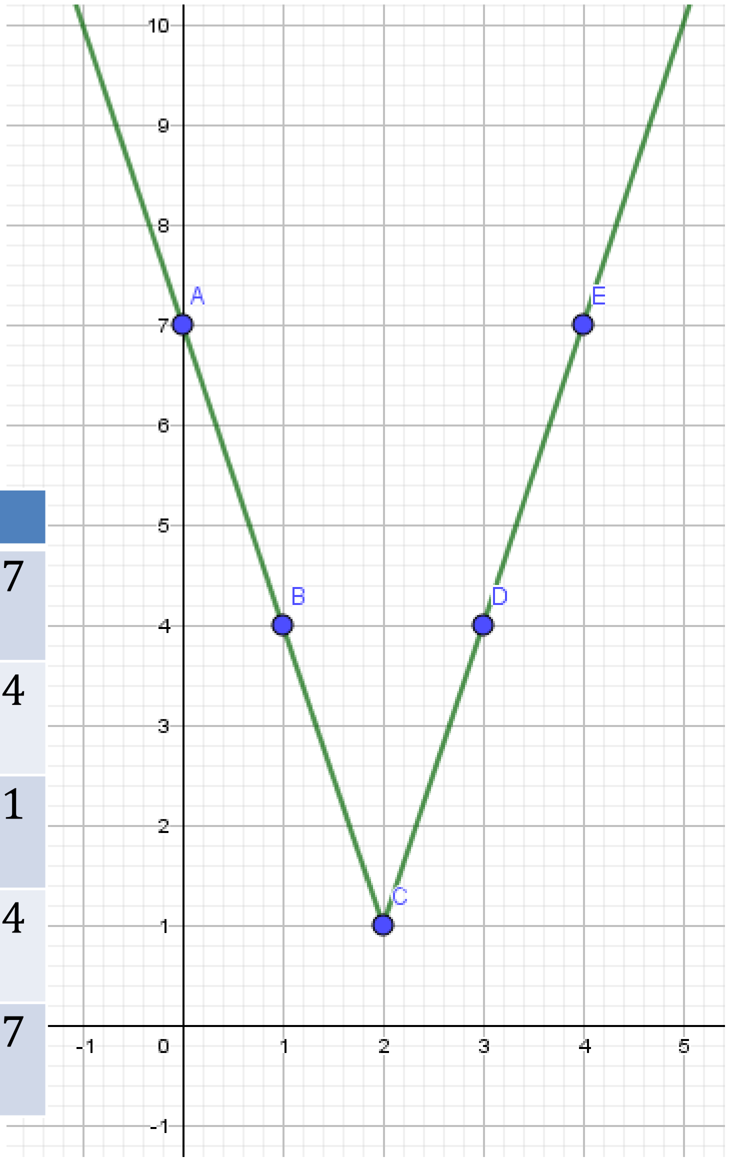
New x values:

$\frac{x}{b} - c \rightarrow \frac{x}{1} - (-2)$

New y values:

$(y \times a) + d \rightarrow 3y + 1$

New x	$g(x)$
$\frac{-2}{1} + 2 = 0$	$3(2) + 1 = 7$
$\frac{-1}{1} + 2 = 1$	$3(1) + 1 = 4$
$\frac{0}{1} + 2 = 2$	$3(0) + 1 = 1$
$\frac{1}{1} + 2 = 3$	$3(1) + 1 = 4$
$\frac{2}{1} + 2 = 4$	$3(2) + 1 = 7$



$\therefore D = (-\infty, \infty)$

$R = [1, \infty)$